Forbidden pairs and the existence of a dominating cycle

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Abstract

A cycle C in a graph G is called dominating if every edge of G is incident with a vertex of C. For a set $\mathcal{H}$ of connected graphs, a graph G is said to be $\mathcal{H}$-free if G does not contain any member of $\mathcal{H}$ as an induced subgraph. When $|\mathcal{H}| = 2$, $\mathcal{H}$ is called a forbidden pair. In this paper, we investigate the set $\boldsymbol{H}$ of pairs $\mathcal{H}$ of connected graphs which satisfies that every 2-connected $\mathcal{H}$-free graph has a dominating cycle. In particular, we show that $\boldsymbol{H}$ is a very small class of pairs of graphs and find some pairs of graphs which belong to $\boldsymbol{H}$.

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1 Introduction

A cycle C in a graph G is called *dominating* if every edge of G is incident with a vertex of C. In this paper, we investigate forbidden subgraphs which imply the existence of a dominating cycle. The origin of our research goes back to results on forbidden subgraphs implying the existence of a Hamilton cycle.

All graphs considered here are finite simple graphs. For standard graph-theoretic terminology not explained in this paper, we refer the readers to [7]. A graph G is said to be *Hamiltonian* if G has a *Hamilton cycle*, i.e., a cycle containing all vertices of G. The study on a Hamilton cycle is one of the most important and basic topics in graph theory. It is known that the problem of determining whether a given graph is Hamiltonian or not belongs to the class of $\boldsymbol{NP}$-complete problems, that is, a difficult problem in a combinatorial sense. So, many researchers

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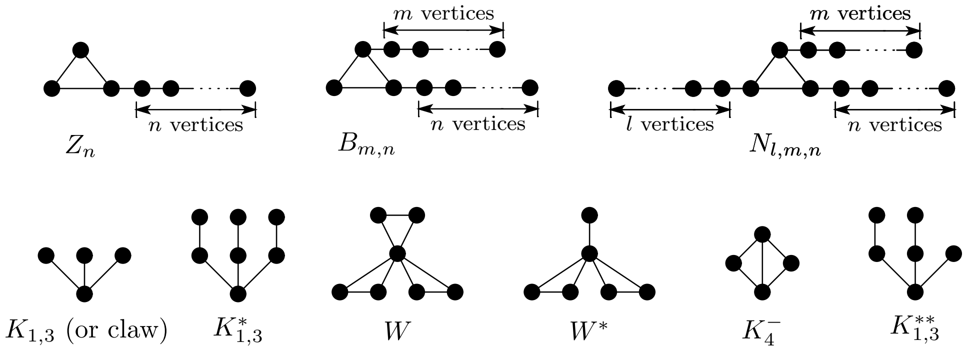

Figure 1: The forbidden subgraphs

have studied sufficient conditions for Hamiltonicity of graphs, and there is a large amount of literature concerning conditions in terms of order, size, vertex degrees, independence number, forbidden subgraphs and so on (see a survey [2]).

Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be *$\mathcal{H}$-free* if G does not contain H as an induced subgraph for all H in $\mathcal{H}$, and we call each graph H of $\mathcal{H}$ a *forbidden subgraph.* If $\mathcal{H} = \{H\}$, then we simply say that G is H-free. We call $\mathcal{H}$ a *forbidden pair* if $|\mathcal{H}| = 2$. When we consider $\mathcal{H}$-free graphs, we assume that each member of $\mathcal{H}$ has order at least 3 because K_2 is the only connected graph of order 2 and connected K_2-free graphs are only K_1 (here K_n denotes the complete graph of order n). In order to state results on forbidden subgraphs clearly, we further introduce several notations. For two graphs H_1 and H_2, we write $H_1 \prec H_2$ if H_1 is an induced subgraph of H_2, and for two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \leq \mathcal{H}_2$ if for every graph H_2 in $\mathcal{H}_2$, there exists a graph H_1 in $\mathcal{H}_1$ with $H_1 \prec H_2$. By the definition of the relation "$\leq$", if $\mathcal{H}_1 \leq \mathcal{H}_2$, then $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

The forbidden pairs that force the existence of a Hamilton cycle in 2-connected graphs had been studied in [5, 8, 16]. Eventually, a characterization of such pairs was accomplished in [1] as follows (here let P_n denote the path of order n, and the graphs $K_{1,3}$ (or claw), $B_{m,n}$ and $N_{l,m,n}$ are the ones that are depicted in Figure 1).

Theorem A (Bedrossian [1]) *Let $\mathcal{H}$ be a set of two connected graphs. Then every 2-connected $\mathcal{H}$-free graph is Hamiltonian if and only if $\mathcal{H} \leq \{K_{1,3}, P_6\}$, $\mathcal{H} \leq \{K_{1,3}, B_{1,2}\}$, or $\mathcal{H} \leq \{K_{1,3}, N_{1,1,1}\}$.*

On the other hand, Faudree, Gould, Ryjáček and Schiermeyer [10] proved that every 2-connected $\{K_{1,3}, Z_3\}$-free graph of order at least 10 is Hamiltonian (here Z_n is the one that is depicted in Figure 1). In [12], the forbidden pairs for Hamiltonicity of 2-connected graphs have been completely determined even when we allow a finite number of exceptions.

Theorem B (Faudree and Gould [12]) *Let $\mathcal{H}$ be a set of two connected graphs. Then every 2-connected $\mathcal{H}$-free graph of sufficiently large order is Hamiltonian if and only if $\mathcal{H} \leq \{K_{1,3}, P_6\}$,*

$\mathcal{H} \leq \{K_{1,3}, Z_3\}$, $\mathcal{H} \leq \{K_{1,3}, B_{1,2}\}$, *or* $\mathcal{H} \leq \{K_{1,3}, N_{1,1,1}\}$.

A 2-*factor* of a graph G is a spanning subgraph of G in which every component is a cycle. It is known that a 2-factor is one of the relaxed structures of a Hamilton cycle since a Hamilton cycle is a connected 2-factor. In fact, the sufficient conditions for the existence of a 2-factor have been extensively studied in order to investigate the difference between the existence of a Hamilton cycle and a 2-factor in graphs (see a survey [15]). As part of it, the forbidden pairs that imply a 2-connected graph has a 2-factor was characterized by J.R. Faudree, R.J. Faudree and Ryjáček [11].

Theorem C (J.R. Faudree, R.J. Faudree and Ryjáček [11]) *Let $\mathcal{H}$ be a set of two connected graphs. Then the following hold.*

(i) *Every 2-connected $\mathcal{H}$-free graph has a 2-factor if and only if $\mathcal{H} \leq \{K_{1,3}, P_7\}$, $\mathcal{H} \leq \{K_{1,3}, Z_4\}$, $\mathcal{H} \leq \{K_{1,3}, B_{1,3}\}$, or $\mathcal{H} \leq \{K_{1,3}, N_{1,1,2}\}$.*

(ii) *Every 2-connected $\mathcal{H}$-free graph of sufficiently large order has a 2-factor if and only if $\mathcal{H} \leq \{K_{1,3}, P_7\}$, $\mathcal{H} \leq \{K_{1,3}, Z_4\}$ $\mathcal{H} \leq \{K_{1,3}, B_{1,4}\}$, $\mathcal{H} \leq \{K_{1,3}, N_{1,1,3}\}$, or $\mathcal{H} \leq \{K_{1,4}, P_4\}$.*

On the other hand, one often try to find a dominating cycle in order to find a Hamilton cycle in a given graph (recall that a cycle C in a graph G is dominating if every edge of G is incident with a vertex of C). For example, if some longest cycle in a graph G is dominating and the independence number of G is at most its minimum degree, then G has a Hamilton cycle (the related results can be found in [4, 24]). It is also shown that the dominating cycle conjecture that "every cyclically 4-edge-connected cubic graph has a dominating cycle" by Fleischner [13] is equivalent to not only the well-known conjecture that "every 4-connected $K_{1,3}$-free graph is Hamiltonian" by Matthews and Sumner [19] but also many other statements on Hamiltonicity of graphs (see a survey [3]). In this sense, a topic on a dominating cycle is one of important relaxations of a Hamilton cycle.

In this paper, our motivation is to investigate the difference between the existence of a Hamilton cycle and a dominating cycle of a 2-connected graph in terms of the forbidden pair.

Problem 1 Determine the set $\boldsymbol{H}$ (resp., $\boldsymbol{H}'$) of pairs $\mathcal{H}$ of connected graphs which satisfy that every 2-connected $\mathcal{H}$-free graph (resp., every 2-connected $\mathcal{H}$-free graph of sufficiently large order) has a dominating cycle.

Concerning the above problem, we first show that $\boldsymbol{H}$ and $\boldsymbol{H}'$ are very small classes of pairs. Let $K^*_{1,3}$, W, W^* and K_4^- be the ones that are depicted in Figure 1, and set $\mathcal{H}_1 = \{K_{1,3}, Z_4\}$, $\mathcal{H}_2 = \{K_{1,3}, B_{1,2}\}$, $\mathcal{H}_3 = \{K_{1,3}, N_{1,1,1}\}$, $\mathcal{H}_4 = \{P_4, W\}$, $\mathcal{H}_5 = \{K^*_{1,3}, Z_1\}$, $\mathcal{H}_6 = \{P_5, W^*\}$ and $\mathcal{H}_7 = \{P_5, K_4^-\}$.

Theorem 1 *Let $\mathcal{H}$ be a set of two connected graphs. If there exists a positive integer $n_0 = n_0(\mathcal{H})$ such that every 2-connected $\mathcal{H}$-free graph of order at least n_0 has a dominating cycle, then $\mathcal{H} \le \mathcal{H}_i$ for some i with $1 \le i \le 7$.*

Theorem 1 implies that $\boldsymbol{H} \subseteq \boldsymbol{H}' \subseteq \{\mathcal{H} : |\mathcal{H}| = 2, \mathcal{H} \le \mathcal{H}_i$ for some i with $1 \le i \le 7\} =: \boldsymbol{H}^*$ So, the remaining problem is that whether $\mathcal{H} \in \boldsymbol{H}$ or not ($\mathcal{H} \in \boldsymbol{H}'$ or not) when $\mathcal{H}$ is a member of $\boldsymbol{H}^*$. We actually guess that the contrary of Theorem 1 holds.

Conjecture 2 *Let $\mathcal{H}$ be a set of two connected graphs. If $\mathcal{H} \in \{\mathcal{H}_i : 1 \le i \le 7\}$, then every 2-connected $\mathcal{H}$-free graph (of sufficiently large order) has a dominating cycle.*

As a partial solution of Conjecture 2, in this article, we further show that $\mathcal{H}_i$ is a member of $\boldsymbol{H}$ ($\subseteq \boldsymbol{H}'$) for $1 \le i \le 4$ and that a pair $\mathcal{H}$ of connected graphs with $\mathcal{H} \le \mathcal{H}_5$ and $\mathcal{H} \ne \mathcal{H}_5$ is also a member of $\boldsymbol{H}$ ($\subseteq \boldsymbol{H}'$) (here $K_{1,3}^{**}$ is the graph obtained from $K_{1,3}^{*}$ by deleting one leaf (see Figure 1) and $\mathcal{H}_5' = \{K_{1,3}^{**}, Z_1\}$).

Theorem 3 *Let $\mathcal{H}$ be a set of two connected graphs. If $\mathcal{H} \in \big\{\mathcal{H}_i : 1 \le i \le 4\big\} \cup \{\mathcal{H}_5'\}$, then every 2-connected $\mathcal{H}$-free graph has a dominating cycle.*

We prove Theorem 1 in Section 3, and slightly stronger statements than Theorem 3 in Section 4 (see Theorems 4–6).

Remark 1 By observing Theorems A, B and C, one may think that we always need an induced subgraph of a star in forbidden pairs for Hamiltonicity-like properties of graphs. In fact, as one of the approach to attack Matthews-Sumner conjecture, the forbidden pair containing $K_{1,3}$ for the existence of a Hamilton cycle in k-connected graphs ($k \ge 3$) have been also studied, e.g., see [14, 17, 18, 21]. However, when we consider the existence of a dominating cycle, the situation is a bit different from Theorems A, B and C, i.e., there exist forbidden pairs which contain no star and force the existence of a dominating cycle in 2-connected graphs (see Theorem 3).

2 Terminology and notation

In this section, we prepare terminology and notation which we use in subsequent sections.

Let G be a graph. We denote by $V(G)$, $E(G)$ and $\Delta(G)$ the vertex set, the edge set and the maximum degree of G, respectively. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph induced by X in G, and let $G - X = G[V(G) \setminus X]$. Let v be a vertex of G. We denote by $N_G(v)$ and $d_G(v)$ the neighborhood and the degree of v in G, respectively. For $X \subseteq V(G) \setminus \{v\}$, we let $N_G(v; X) = N_G(v) \cap X$, and for $V, X \subseteq V(G)$ with $V \cap X = \emptyset$, let $N_G(V; X) = \bigcup_{v \in V} N_G(v; X)$. We often identify a subgraph F of G with its vertex set $V(F)$ (for example, $N_G(v; V(F))$ is often denoted by $N_G(v; F)$). For a positive integer l, we define $V_l(G) = \{v \in V(G) : d_G(v) = l\}$. For

$u, v \in V(G)$, $\mathrm{dist}_G(u, v)$ denotes the distance between u and v in G, and we define the *diameter* $\mathrm{diam}(G)$ of G by $\mathrm{diam}(G) = \max\{\mathrm{dist}_G(u, v) : u, v \in V(G)\}$. When G has a cycle, we denote by $c(G)$ the *circumference* of G, i.e., the length of the longest cycle of G. A path with end vertices u and v is denoted by a (u, v)-*path*.

We write a cycle (or a path) C with a given orientation by $\overrightarrow{C}$. If there exists no chance of confusion, we abbreviate $\overrightarrow{C}$ by C. Let $\overrightarrow{C}$ be an oriented cycle or a path. For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$ the (x, y)-path on $\overrightarrow{C}$. The reverse sequence of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. For $u \in V(C)$, we denote the h-th successor and the h-th predecessor of u on $\overrightarrow{C}$ by u^{+h} and u^{-h}, respectively, and let $u^{+0} = u$. For $X \subseteq V(C)$, we define $X^{+h} = \{x^{+h} : x \in X\}$ and $X^{-h} = \{x^{-h} : x \in X\}$, respectively. We abbreviate u^{+1}, u^{-1}, X^{+1} and X^{-1} by u^+, u^-, X^+ and X^-, respectively.

For two graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, let $G_1 \cup G_2$ denote the union of G_1 and G_2, and let $G_1 + G_2$ denote the join of G_1 and G_2, i.e., the graph obtained from $G_1 \cup G_2$ by joining each vertex in $V(G_1)$ to all vertices in $V(G_2)$. For a graph G and $l \geq 1$, let lG denote the union of l vertex-disjoint copies of G.

3 Proof of Theorem 1

We first prepare the following lemma concerning the property of a finite set of forbidden subgraphs that imply the existence of a dominating cycle.

Lemma 1 *Let $\mathcal{H}$ be a finite set of connected graphs, and suppose that there exists a positive integer $n = n(\mathcal{H})$ such that every 2-connected $\mathcal{H}$-free graph of order at least n has a dominating cycle.*

(i) *Then $\mathcal{H}$ contains a tree T with $\Delta(T) \leq 3$ and $|V_3(T)| \leq 1$.*

(ii) *If $|\mathcal{H}| = 2$ and $\mathcal{H}$ contains a graph with diameter at least 3, then the other one is an induced subgraph of $K_1 + 3K_2$, or isomorphic to K_4^-.*

In order to prove Lemma 1, we define the following graphs A_s, A'_s and A''_s (see Figure 2). Note that each of A_s, A'_s and A''_s is 2-connected and contains no dominating cycle.

- For each $s \geq 2$, let A_s be the graph consisting of the union of three internally disjoint P_{s+2}'s that have the same two distinct end vertices.
- For each $s \geq 3$, let $A'_s = 2K_1 + sK_2$.
- For each $s \geq 2$, let $A''_s = K_2 + (2K_2 \cup K_s)$.

Proof of Lemma 1. (i) Let $m = \max\{|V(H)| : H \in \mathcal{H}\}$, and let $n_1 = \max\{n, m\}$. Since A_{n_1} is a 2-connected graph of order at least n_1 ($\geq n$) having no dominating cycle, it follows that there

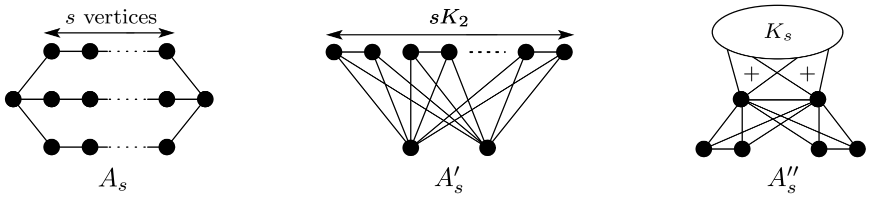

Figure 2: The graphs A_s, A'_s and A''_s

exists a graph H in $\mathcal{H}$ such that $H \prec A_{n_1}$. Observe that, by the definition of A_{n_1}, all cycles in A_{n_1} have $2n_1 + 2$ ($\geq 2m + 2$) vertices and the distance of two vertices with degree 3 in A_{n_1} is $n_1 + 1$ ($\geq m + 1$). These facts imply that H is a tree with $\Delta(H) \leq 3$ and $|V_3(H)| \leq 1$.

(ii) Write $\mathcal{H} = \{H_1, H_2\}$, and assume that $\mathrm{diam}(H_1) \geq 3$. Let $n_2 = \max\{n, 3\}$. Since $\mathrm{diam}(H_1) \geq 3$, we have $P_4 \prec H_1$, and hence A'_{n_2} does not contain H_1 as an induced subgraph because A'_{n_2} contains no P_4 as an induced subgraph. Similarly, we see that A''_{n_2} does not contain H_1 as an induced subgraph. On the other hand, both of A'_{n_2} and A''_{n_2} are 2-connected graphs of order at least n_2 ($\geq n$) having no dominating cycle. This implies that H_2 is a common induced subgraph of A'_{n_2} and A''_{n_2}. Hence it is easy to check that $H_2 \prec K_1 + 3K_2$ or $H_2 \cong K_4^-$. $\square$

We further define five graphs of 2-connected graphs having no dominating cycle as follows (see Figure 3).

- For each $s \geq 2$, let $A_s^{(1)}$ be the graph which consists of two vertex-disjoint triangles connected by three vertex-disjoint paths of orders $s + 2$, respectively.
- For each $s \geq 4$, let G_i ($1 \leq i \leq 3$) be a complete graph of order s, and let $A_s^{(2)}$ be the graph obtained from $G_1 \cup G_2 \cup G_3$ by joining u_i to u_{i+1} and v_i to v_{i+1} for $1 \leq i \leq 3$, where u_i and v_i are distinct two vertices of G_i and $u_4 = u_1, v_4 = v_1$.
- For each $s \geq 4$, let $A_s^{(3)} = A_s^{(2)} - \{u_i v_i : 1 \leq i \leq 3\}$.
- For each $s \geq 2$, let $A_s^{(4)}$ be the graph obtained from $K_2 + (K_2 \cup K_s)$ by subdividing the edge xy twice, where $\{x, y\}$ is the unique 2-cut set of $K_2 + (K_2 \cup K_s)$.
- For each $s \geq 3$, let $A_s^{(5)}$ be the graph defined by $V(A_s^{(5)}) = \{x_i, y_{i,j} : 1 \leq i \leq 2, 1 \leq j \leq s\}$ and $E(A_s^{(5)}) = \{x_1 x_2\} \cup \{y_{1,j} y_{2,j} : 1 \leq j \leq s\} \cup \{x_i y_{i,j} : 1 \leq i \leq 2, 1 \leq j \leq s\}$.

By the definition of $A_s^{(j)}$, we can obtain the following lemma. (Since the proof is easy, we omit it.)

Lemma 2 (i) *$A_s^{(1)}$, $A_s^{(2)}$ and $A_s^{(3)}$ are $K_{1,3}$-free graphs. Furthermore, $A_s^{(1)}$ is K_4^--free.*

(ii) *Every connected induced subgraph of $A_s^{(1)}$ with at most s vertices is also an induced subgraph of $N_{i,j,k}$ for some integers i, j and k. In particular, every induced subtree of $A_s^{(1)}$ with at most s vertices is a path.*

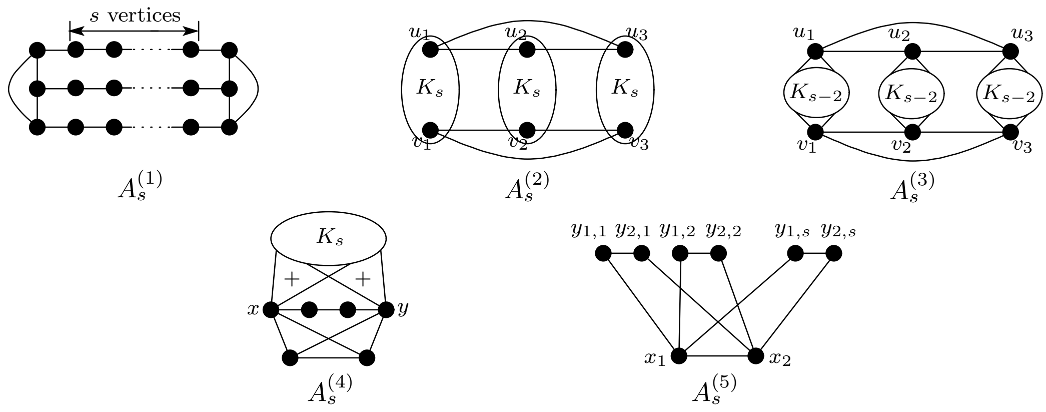

Figure 3: The graph $A_s^{(j)}$

(iii) *Every induced subpath of $A_s^{(2)}$ has at most 6 vertices.*

(iv) *$A_s^{(2)}$ contains neither $N_{1,1,2}$ nor $B_{1,3}$ as an induced subgraph.*

(v) *$A_s^{(3)}$ contains no $B_{2,2}$ as an induced subgraph.*

(vi) *$A_s^{(4)}$ is P_5-free and $A_s^{(5)}$ is K_3-free.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let $\mathcal{H}$ be a set of two connected graphs, and suppose that there exists a positive integer $n_0 = n_0(\mathcal{H})$ such that every 2-connected $\mathcal{H}$-free graph of order at least n_0 has a dominating cycle. We show that $\mathcal{H} \leq \mathcal{H}_i$ for some i with $1 \leq i \leq 7$.

If $\mathcal{H}$ contains P_3, then $\mathcal{H} \leq \mathcal{H}_1$. Thus we may assume that $\mathcal{H}$ does not contain P_3. Write $\mathcal{H} = \{H_1, H_2\}$, and let $n = \max\{n_0, 4, |V(H_1)|, |V(H_2)|\}$. Then for each j with $1 \leq j \leq 5$,

$$H_1 \prec A_n^{(j)} \text{ or } H_2 \prec A_n^{(j)} \tag{3.1}$$

because $A_n^{(j)}$ is a 2-connected graph of order at least n ($\geq n_0$) having no dominating cycle.

We divide the proof into two cases according as $\mathcal{H}$ contains $K_{1,3}$ or not.

Case 1. H_i is isomorphic to $K_{1,3}$ for some i with $i \in \{1, 2\}$.

We may assume that $H_1 \cong K_{1,3}$. Then it follows from Lemma 2 (i) and (3.1) that $H_2 \prec A_n^{(j)}$ for each $1 \leq j \leq 3$, i.e., H_2 is a common induced subgraph of $A_n^{(1)}$, $A_n^{(2)}$ and $A_n^{(3)}$. Since $n \geq |V(H_2)|$, it follows from Lemma 2 (ii) that H_2 is an induced subgraph of $N_{i,j,k}$ for some integers i, j and k. This together with Lemma 2 (iii) implies that H_2 is an induced subgraph of either Z_4, $B_{1,3}$ or $N_{2,2,2}$. If $H_2 \prec Z_4$, then $\mathcal{H} \leq \mathcal{H}_1$; if $H_2 \prec B_{1,3}$, then by Lemma 2 (iv), either $H_2 \prec P_6$, $H_2 \prec Z_3$ or $H_2 \prec B_{1,2}$, and hence $\mathcal{H} \leq \mathcal{H}_1$ or $\mathcal{H} \leq \mathcal{H}_2$; if $H_2 \prec N_{2,2,2}$, then by Lemma 2 (iv) and (v), either $H_2 \prec P_6$, $H_2 \prec Z_2$, $H_2 \prec B_{1,2}$ or $H_2 \prec N_{1,1,1}$, and hence $\mathcal{H} \leq \mathcal{H}_1$, $\mathcal{H} \leq \mathcal{H}_2$ or $\mathcal{H} \leq \mathcal{H}_3$.

Case 2. H_i is not isomorphic to $K_{1,3}$ for $i \in \{1, 2\}$.

By Lemma 1 (i), we may assume that H_1 is a tree with $\Delta(H_1) \leq 3$ and $|V_3(H_1)| \leq 1$. Since $H_1 \ncong P_3$ and $H_1 \ncong K_{1,3}$ by the assumption of Case 2, we have $\mathrm{diam}(H_1) \geq 3$. Hence by Lemma 1 (ii), H_2 is an induced subgraph of $K_1 + 3K_2$ or isomorphic to K_4^-. In particular, this implies that H_2 has a triangle (note that if H_2 is a tree, then either $H_2 \cong K_{1,3}$ or $H_2 \cong P_3$, a contradiction). Therefore, it follows from Lemma 2 (vi) and (3.1) that $H_1 \prec A_n^{(5)}$. Combining this with the fact that $\Delta(H_1) \leq 3$ and $|V_3(H_1)| \leq 1$, we have $H_1 \prec K_{1,3}^*$.

We divide the proof of Case 2 into three cases according as $H_2 \prec K_1 + 3K_2$ and H_1 is not a path, $H_2 \prec K_1 + 3K_2$ and H_1 is a path, or $H_2 \cong K_4^-$.

Subcase 2.1. H_2 is an induced subgraph of $K_1 + 3K_2$ and H_1 is not a path.

Since H_1 is a tree which is not a path, H_1 is not an induced subgraph of $A_n^{(1)}$ by Lemma 2 (ii). This together with (3.1) implies that $H_2 \prec A_n^{(1)}$. Combining this with the assumption that H_2 is an induced subgraph of $K_1 + 3K_2$, we see that $H_2 \prec Z_1$. Since $H_1 \prec K_{1,3}^*$, $\mathcal{H} \leq \mathcal{H}_5$.

Subcase 2.2. H_2 is an induced subgraph of $K_1 + 3K_2$ and H_1 is a path.

If H_1 is a path of order at most 4, then $\mathcal{H} \leq \mathcal{H}_4$ (note that $K_1 + 3K_2 \cong W$). Thus we may assume that H_1 is a path of order 5 because $H_1 \prec K_{1,3}^*$. Then by Lemma 2 (vi) and (3.1), $H_2 \prec A_n^{(4)}$. Since $H_2 \prec K_1 + 3K_2$ and $A_n^{(4)}$ contains no $K_1 + 3K_2$ as an induced subgraph, H_2 is an induced subgraph of $K_1 + (K_1 \cup 2K_2)$. Thus $\mathcal{H} \leq \mathcal{H}_6$ (note that $K_1 + (K_1 \cup 2K_2) \cong W^*$).

Subcase 2.3. H_2 is isomorphic to K_4^-.

Then by Lemma 2 (i), (3.1) and the assumption of Subcase 2.3, we have $H_1 \prec A_n^{(1)}$. Since $H_1 \prec K_{1,3}^*$, H_1 is a path of order at most 5 by Lemma 2 (ii). Thus $\mathcal{H} \leq \mathcal{H}_7$.

This completes the proof of Theorem 1. $\square$

4 Proof of Theorem 3

In this section, we prove Theorem 3. To prove this, we show that the following theorems hold, which immediately imply Theorem 3 (note that we actually prove slightly stronger statements.)

Theorem 4 *If $\mathcal{H} \in \{\mathcal{H}_i : 1 \leq i \leq 3\}$, then every longest cycle of a 2-connected $\mathcal{H}$-free graph is a dominating cycle of the graph.*

Theorem 5 *Every longest cycle of a 2-connected $\{P_4, W\}$-free graph is a dominating cycle of the graph.*

Theorem 6 *A longest cycle of a 2-connected $\{K_{1,3}^{**}, Z_1\}$-free graph is a dominating cycle of the graph.*

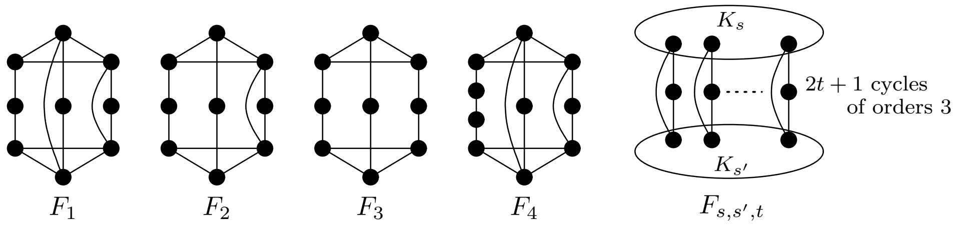

Figure 4: The graph F_i and $F_{s,s',t}$

We will prove Theorems 4 and 5 in Subsections 4.1 and 4.2, respectively, and we will prove Theorem 6 in Subsections 4.3–4.5.

4.1 $K_{1,3}$-free graphs

In this subsection, we prove Theorem 4. In order to prove it, we use some concepts and known results.

In [22], Ryjáček introduced the concept of a closure for claw-free graphs as follows. Let G be a claw-free graph. For each vertex v of G, $G[N_G(v)]$ has at most two components; otherwise G contains a $K_{1,3}$ as an induced subgraph. If $G[N_G(v)]$ has two components, both of them must be cliques. In the case that $G[N_G(v)]$ is connected, we add edges joining all pairs of nonadjacent vertices in $N_G(v)$. The *closure* $\mathrm{cl}(G)$ of G is a graph obtained by recursively repeating this operation, as long as this is possible. In [22], it was shown that the closure of a graph has the following property.

Theorem D (Ryjáček [22]) *If G is a claw-free graph, then the following hold.*

(i) $\mathrm{cl}(G)$ *is well-defined, (i.e., uniquely defined).*

(ii) $c(G) = c(\mathrm{cl}(G))$.

On the other hand, Brousek, Ryjáček and Favaron [6] characterized 2-connected $\{K_{1,3}, Z_4\}$-free graphs having no Hamilton cycle. Let F_1, F_2, F_3 and F_4 be the ones that are depicted in Figure 4, and set $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$. For each s, s' and t with $s' \geq s \geq 3$ and $1 \leq t \leq (s-1)/2$, let $F_{s,s',t}$ be the graph which consists of vertex-disjoint K_s and $K_{s'}$ connected by $2t+1$ vertex-disjoint cycles of orders 3, respectively (see Figure 4), and set $\mathcal{F}' = \{F_{s,s',t} : s' \geq s \geq 3, 1 \leq t \leq (s-1)/2\}$.

Theorem E (Brousek, Ryjáček and Favaron [6]) *Let G be a 2-connected $\{K_{1,3}, Z_4\}$-free graph. If G is not Hamiltonian, then $G \in \mathcal{F}$ or $\mathrm{cl}(G) \in \mathcal{F}'$.*

Now we prove Theorem 4.

Proof of Theorem 4. Let $\mathcal{H} \in \{\mathcal{H}_i : 1 \leq i \leq 3\}$, and let G be a 2-connected $\mathcal{H}$-free graph. We show that every longest cycle of G is a dominating cycle of G. If G is Hamiltonian, then the

assertion clearly holds; thus we may assume that G is not Hamiltonian. Then by Theorem A, $\mathcal{H} \neq \mathcal{H}_2$ ($= \{K_{1,3}, B_{1,2}\}$) and $\mathcal{H} \neq \mathcal{H}_3$ ($= \{K_{1,3}, N_{1,1,1}\}$). Thus $\mathcal{H} = \mathcal{H}_1$ ($= \{K_{1,3}, Z_4\}$). Then it follows from Theorem E that $G \in \mathcal{F}$ or $\mathrm{cl}(G) \in \mathcal{F}'$. Since each graph F in $\mathcal{F} \cup \mathcal{F}'$ has a longest cycle of order $|V(F)| - 1$, this together with Theorem D (ii) implies that the longest cycle of G has $|V(G)| - 1$ vertices; thus every longest cycle of G is a dominating cycle. $\square$

4.2 P_4-free graphs

In this subsection, we prove Theorem 5. To prove this, we use the following lemma concerning the property of P_4-free graphs. (In [23], a theorem which implies Lemma A was proved by Seinseche, and also see [9, 11].)

Lemma A *Let G be a P_4-free graph. If G is k-connected and $|V(G)| \geq 2k$, then there exists a partition $\{A, B\}$ of $V(G)$ with $|A| \geq k$ and $|B| \geq k$ such that every vertex in A is adjacent to each vertex in B.*

Now we prove Theorem 5.

Proof of Theorem 5. Let G be a 2-connected $\{P_4, W\}$-free graph. We may assume that $|V(G)| \geq 4$ (otherwise, the assertion clearly holds). Then by applying Lemma A as $k = 2$, there exists a partition $\{A, B\}$ of $V(G)$ with $|A| \geq 2$ and $|B| \geq 2$ such that every vertex in A is adjacent to all vertices in B. By symmetry, we may assume that $|A| \geq |B|$. Suppose that G has a longest cycle $\overrightarrow{C}$ which is not a dominating cycle of G.

Claim 1 *$B \subseteq V(C)$.*

Proof. Suppose that $B \not\subseteq V(C)$, and let $u \in B \setminus V(C)$. Since $G[A' \cup B]$ is Hamiltonian for $A' \subseteq A$ with $|A'| = |B|$, we have $|V(C)| = c(G) \geq 2|B|$. Hence $|V(C) \cap A| = |V(C) \setminus B| = |V(C) \setminus (B \setminus \{u\})| \geq |V(C)| - (|B| - 1) \geq 2|B| - (|B| - 1) = |B| + 1$. Since $C - B$ ($= C - (B \setminus \{u\})$) has at most $|B| - 1$ components, this implies that there exists a vertex x of C such that $x, x^+ \in A$. Then the cycle $x^+ \overrightarrow{C} x u x^+$ is a longer cycle than C, a contradiction. $\square$

Since C is not a dominating cycle of G, it follows from Claim 1 that there exist vertices $x_1, x_2 \in A \setminus V(C)$ with $x_1 x_2 \in E(G)$. Moreover, by again Claim 1 and since $|B| \geq 2$, we can take distinct two vertices u_1 and u_2 in $B \cap V(C)$.

Claim 2 $N_G(\{x_1, x_2\}; \{u_1^+, u_1^{+2}, u_2^+, u_2^{+2}\}) = \emptyset$. *In particular,* $\{u_1^+, u_1^{+2}, u_2^+, u_2^{+2}\} \subseteq A$.

Proof. If there exists a vertex u in $N_G(x_1; \{u_1^+, u_1^{+2}\})$, then $x_1 u \overrightarrow{C} u_1 x_2 x_1$ is a cycle which contains $(V(C) \setminus \{u_1^+, u_1^{+2}\}) \cup \{u, x_1, x_2\}$. This contradicts the maximality of $|V(C)|$. Thus $N_G(x_1, \{u_1^+, u_1^{+2}\}) = \emptyset$. By the symmetry of u_1 and u_2, we also have $N_G(x_1, \{u_2^+; u_2^{+2}\}) = \emptyset$,

and hence by the symmetry of x_1 and x_2, we have $N_G(x_2, \{u_1^+, u_1^{+2}, u_2^+, u_2^{+2}\}) = \emptyset$. In particular, since $\{x_1, x_2\} \subseteq A$, this implies that $\{u_1^+, u_1^{+2}, u_2^+, u_2^{+2}\} \subseteq A$. $\square$

Since $u_1 \in B$ and $\{x_1, x_2, u_1^+, u_1^{+2}, u_2^+, u_2^{+2}\} \subseteq A$ by Claim 2, we have that $G[\{u_1, x_1, x_2, u_1^+, u_1^{+2}, u_2^+, u_2^{+2}\}]$ contains W as a subgraph. Since G is W-free, Claim 2 yields that $N_G(\{u_1^+, u_1^{+2}\}; \{u_2^+, u_2^{+2}\}) \neq \emptyset$. If $u_1^+ u_2^+ \in E(G)$, then $u_1^+ u_2^+ \overrightarrow{C} u_1 x_1 u_2 \overleftarrow{C} u_1^+$ is a longer cycle than C, a contradiction. If $u_1^+ u_2^{+2} \in E(G)$, then $u_1^+ u_2^{+2} \overrightarrow{C} u_1 x_1 x_2 u_2 \overleftarrow{C} u_1^+$ is a longer cycle than C, a contradiction again. Similarly, we have $u_1^{+2} u_2^+ \notin E(G)$, and hence we have $u_1^{+2} u_2^{+2} \in E(G)$. But, then $u_1^+ u_1^{+2} u_2^{+2} u_2^+$ is an induced path of G, which contradicts the assumption that G is P_4-free.

This completes the proof of Theorem 5. $\square$

4.3 Proof of Theorem 6

The proof of Theorem 6 is actually divided into two parts according as the graph contains a triangle or not. To do that, we use the following.

Lemma B (Olariu [20]) *Let G be a connected Z_1-free graph. If G contains a triangle, then G is a complete multipartite graph.*

Theorem 7 *A longest cycle of a 2-connected $\{K_{1,3}^{**}, K_3\}$-free graph is a dominating cycle of the graph.*

Here we prove Theorem 6 assuming Theorem 7. We will show Theorem 7 in Subsections 4.4 and 4.5.

Proof of Theorem 6. Let G be a 2-connected $\{K_{1,3}^{**}, Z_1\}$-free graph. If G is K_3-free, then by Theorem 7, G has a longest cycle which is a dominating cycle. Thus we may assume that G contains a triangle. Then by Lemma B, G is a complete multipartite graph. Let $\overrightarrow{C}$ be a longest of G. Suppose that there exists an edge xy in $G - V(C)$, and let $u \in V(C)$. If some vertex a in $\{x, y\}$ belongs to a different partite set from u and u^+, then $uau^+\overrightarrow{C}u$ is a longer cycle than C, a contradiction. Thus we may assume that u and x belong to the same partite set, and u^+ and y belong to the same partite set (note that u and u^+ belong to different partite sets). Then $uyxu^+\overrightarrow{C}u$ is a longer cycle than C, a contradiction again. Thus C is a dominating cycle of G. $\square$

4.4 Preparation for the proof of Theorem 7

In this subsection, we prepare lemmas which will be used in the proof of Theorem 7.

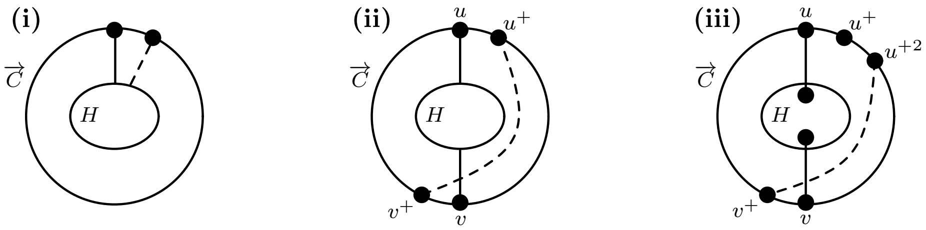

Figure 5: Lemma 3

Now let G be a 2-connected graph. Let $\overrightarrow{C}$ be a longest cycle of G, and let H be a component of $G - V(C)$. (Note that $|V(C)| \geq 4$.) Then by the maximality of $|V(C)|$, we can easily obtain the following lemma.

Lemma 3 (i) *$N_G(H; C) \cap N_G(H; C)^+ = \emptyset$ (see the left of Figure 5).*

(ii) *If u and v are distinct two vertices in $N_G(H; C)$, then $E(G) \cap \{u^+v^+, u^-v^-\} = \emptyset$ (see the center of Figure 5).*

(iii) *If u and v are distinct two vertices in $N_G(H; C)$ such that $|N_G(u; H) \cup N_G(v; H)| \geq 2$, then $E(G) \cap \{u^{+2}v^+, u^+v^{+2}, u^{-2}v^-, u^-v^{-2}\} = \emptyset$. In particular, $u^{+2} \neq v$ and $u^{-2} \neq v^-$ (see the right of Figure 5).*

Moreover, we give the following lemma concerning $\{K_{1,3}^{**}, K_3\}$-free graphs.

Lemma 4 *Let $x \in V(H)$ and $u \in N_G(x; C)$. If G is $\{K_{1,3}^{**}, K_3\}$-free and $|V(H)| \geq 2$, then $E(G) \cap \{u^{+2}u^-, u^{+2}x\} \neq \emptyset$ and $E(G) \cap \{u^{-2}u^+, u^{-2}x\} \neq \emptyset$.*

Proof of Lemma 4. Since $|V(H)| \geq 2$, $N_H(x) \neq \emptyset$. Let $x' \in N_H(x)$. Then by Lemma 3 (i), $E(G) \cap \{u^+x, u^-x, u^+x', u^-x'\} = \emptyset$. By Lemma 3 (iii), $u^{+2}x' \notin E(G)$. Moreover, since G is K_3-free, $E(G) \cap \{uu^{+2}, u^+u^-, ux'\} = \emptyset$. Therefore, if $E(G) \cap \{u^{+2}u^-, u^{+2}x\} = \emptyset$, then $G[\{u, u^+, u^{+2}, u^-, x, x'\}]$ is isomorphic to $K_{1,3}^{**}$, a contradiction. Thus $E(G) \cap \{u^{+2}u^-, u^{+2}x\} \neq \emptyset$. By the symmetry of $\overrightarrow{C}$ and $\overleftarrow{C}$, we have that $E(G) \cap \{u^{-2}u^+, u^{-2}x\} \neq \emptyset$. $\square$

4.5 Proof of Theorem 7

In this subsection, we prove Theorem 7.

Proof of Theorem 7. Let G be a 2-connected $\{K_{1,3}^{**}, K_3\}$-free graph, and we show that G has a longest cycle which is a dominating cycle of G. By way of a contradiction, suppose that every longest cycle of G is not a dominating cycle of G. For a cycle C of G, let $\mu(C) = \max\{|V(H)| : H$ is a component of $G - V(C)\}$. Then $\mu(C) \geq 2$ for every longest cycle C of G. For a cycle C of

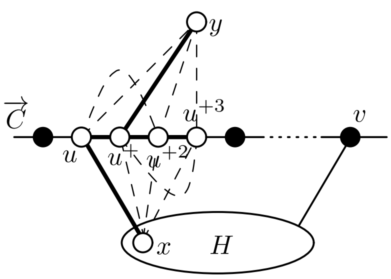

Figure 6: Claim 3

G, we further define $\omega(C) = |\{H : H \text{ is a component of } G - V(C) \text{ such that } |V(H)| = \mu(C)\}|$. Let $\overrightarrow{C}$ be a longest cycle of G. We choose C so that

(C1) $\mu(C)$ is as small as possible, and

(C2) $\omega(C)$ is as small as possible, subject to (C1).

Let H be a component of $G - V(C)$ such that $|V(H)| = \mu(C)$ (≥ 2). Since G is 2-connected, there exist distinct two vertices u and v in C such that $N_G(u; H) \neq \emptyset$, $N_G(v; H) \neq \emptyset$, $|N_G(u; H) \cup N_G(v; H)| \geq 2$ and $N_G(H; u^+\overrightarrow{C}v^-) = \emptyset$. We choose the longest cycle C of G, the component H of $G - V(C)$ with $|V(H)| = \mu(C)$, the vertices u and v so that

(C3) $|V(u^+\overrightarrow{C}v^-)|$ is as large as possible, subject to (C1) and (C2).

By Lemma 3 (i) and (iii), $|V(u\overrightarrow{C}v)| \geq 4$ and $|V(v\overrightarrow{C}u)| \geq 4$. Since $N_G(H; u^+\overrightarrow{C}v^-) = \emptyset$, it follows from Lemma 4 that $u^{+2}u^- \in E(G)$, and hence by Lemma 3 (ii) and (iii), $|V(u\overrightarrow{C}v)| \geq 6$.

Claim 3 *$yu^{+3} \in E(G)$ for $y \in N_G(u^+; G - V(C))$.*

Proof. Suppose that $yu^{+3} \notin E(G)$ for some vertex $y \in N_G(u^+; G - V(C))$. By the choice of u and v, $y \notin V(H)$. Let $x \in N_G(u; H)$. Since G is K_3-free, $E(G) \cap \{yu, yu^{+2}, uu^{+2}, u^+u^{+3}\} = \emptyset$. Since $N_G(H; u^+\overrightarrow{C}v^-) = \emptyset$, $yu^{+3} \notin E(G)$ and $G[\{x, y, u, u^+, u^{+2}, u^{+3}\}] \not\cong K_{1,3}^{**}$, these imply that $uu^{+3} \in E(G)$ (see Figure 6). However, $G[\{x, x', y, u, u^+, u^{+3}\}]$ is isomorphic to $K_{1,3}^{**}$ where $x' \in N_H(x)$ because $N_G(H; u^+\overrightarrow{C}v^-) = \emptyset$ and G is K_3-free, a contradiction. □

Claim 4 $N_G(u^{-2}; H) = \emptyset$.

Proof. Suppose that $N_G(u^{-2}; H) \neq \emptyset$, and let $x \in N_G(u; H)$. By Lemma 3 (iii), $N_G(u^{-2}; H) = \{x\}$. If there exists a vertex $y \in N_G(u^+; G - V(C))$, then by Claim 3, $yu^{+3} \in E(G)$, and hence $u^{-2}xuu^-u^{+2}u^+yu^{+3}\overrightarrow{C}u^{-2}$ is a longer cycle than C (note that by the choice of u and v, $y \notin V(H)$), a contradiction. Thus $N_G(u^+; G - V(C)) = \emptyset$. Then $D := u^{-2}xuu^-u^{+2}\overrightarrow{C}u^{-2}$ is a cycle in G such that $V(D) = (V(C) \setminus \{u^+\}) \cup \{x\}$. Since $N_G(u^+; G - V(C)) = \emptyset$, it follows

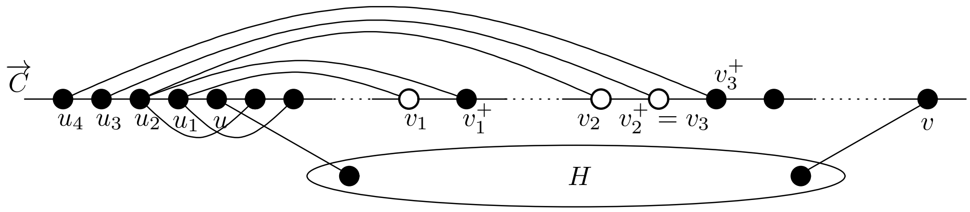

Figure 7: The insertible path $u_1u_2u_3$ of C

that u^+ is a component of $G - V(D)$, and $G - V(D)$ contains some components whose union is $H - \{x\}$, which contradicts the choice (C1) or (C2). $\square$

By Lemma 4 and Claim 4, we have $u^+u^{-2} \in E(G)$, and hence by Lemma 3 (ii) and (iii), $|V(v\overrightarrow{C}u)| \geq 6$. Moreover, by the maximality of $|V(C)|$, we can easily see that the following holds.

Claim 5 $E(G) \cap \{u^{-3}v^-, u^{-4}v^-\} = \emptyset$.

Proof. Let $x \in N_G(u; H)$ and $x' \in N_G(v; H)$ with $x \neq x'$, and let $\overrightarrow{P}$ be an (x, x')-path in H. If v^- is adjacent to a vertex $z \in \{u^{-3}, u^{-4}\}$, then $zv^-\overleftarrow{C}u^+u^{-2}\overrightarrow{C}ux\overrightarrow{P}x'v\overrightarrow{C}z$ is a longer cycle than C, a contradiction. Thus $E(G) \cap \{u^{-3}v^-, u^{-4}v^-\} = \emptyset$. $\square$

Let $w \in N_G(H; v\overrightarrow{C}u^-)$. We choose w so that $|V(w\overrightarrow{C}u)|$ is as small as possible. Note that by the choice of w, $N_G(H; w^+\overrightarrow{C}u^-) = \emptyset$ (possibly $w = v$). By Lemma 3 (i) and Claim 4, $w \notin \{u^-, u^{-2}\}$. Since $u^{-2}u^+ \in E(G)$, it follows from Lemma 3 (ii) that $w \neq u^{-3}$. Write $u^-\overleftarrow{C}w^+ = u_1u_2 \ldots u_{s-1}u_s$ ($s \geq 3$). For an integer k with $1 \leq k \leq s-1$, we call $u_1\overleftarrow{C}u_k$ ($= u_1u_2 \ldots u_k$) an *insertible path of* C if there exist distinct k vertices $v_1, v_2, \ldots, v_k$ satisfying the following (see Figure 7):

(I1) The vertices $v_1, v_2, \ldots, v_k$ appear in this order along $\overrightarrow{C}$ and $v_i \in V(u^{+2}\overrightarrow{C}v^{-2})$ for each i with $1 \leq i \leq k$.

(I2) $\{u_iv_i, u_{i+1}v_i^+\} \subseteq E(G)$ for each i with $1 \leq i \leq k$.

(I3) If $v_k \neq v^{-2}$, then $N_G(u_i; v_i^+\overrightarrow{C}v^{-2}) = \emptyset$ for each i with $1 \leq i \leq k$.

For an insertible path $u_1\overleftarrow{C}u_k$ of C, the vertices $v_1, v_2, \ldots, v_k$ satisfying the conditions (I1)–(I3) is called *bridging vertices* of $u_1\overleftarrow{C}u_k$.

Claim 6 *Let k and l be integers with $2 \leq k \leq s-1$ and $1 \leq l \leq k+1$. If $u_1\overleftarrow{C}u_{k-1}$ is an insertible path of C, then $u_lv^- \notin E(G)$. In particular, if $v_1, \ldots, v_{k-1}$ are bridging vertices of $u_1\overleftarrow{C}u_{k-1}$, then $v_{k-1} \neq v^{-2}$.*

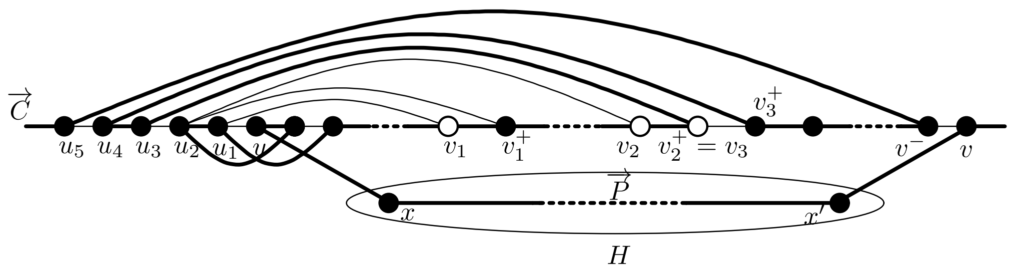

Figure 8: The case of $k-1=3$ and $l=5$

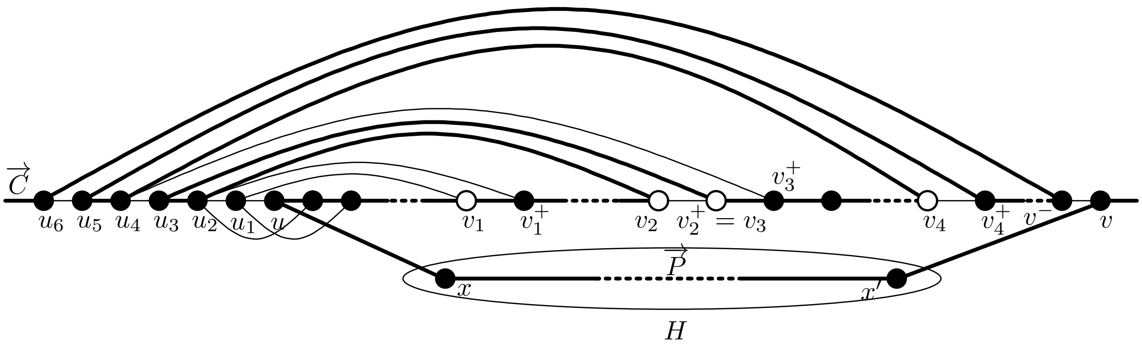

Figure 9: The case of $k-1=4$ and $l=6$

Proof. By Lemma 3 (ii), (iii) and Claim 5, we may assume that $k \geq 4$ and $l \geq 5$. Let $v_1, \ldots, v_{k-1}$ be bridging vertices of $u_1 \overleftarrow{C} u_{k-1}$. Suppose that $u_l v^- \in E(G)$. Let $x \in N_G(u;H)$ and $x' \in N_G(v;H)$ with $x \neq x'$, and let $\overrightarrow{P}$ be an (x,x')-path in H. If l is odd, then by the condition (I2), $D := u_l v^- \overleftarrow{C} v_{l-2}^+ u_{l-1} u_{l-2} v_{l-2} \overleftarrow{C} v_{l-4}^+ u_{l-3}\ u_{l-4} v_{l-4} \overleftarrow{C} \ldots v_3 \overleftarrow{C} u^{+2} u_1 u_2 u^+ u x \overrightarrow{P} x' v \overrightarrow{C} u_l$ is a cycle in G such that $V(D) = V(C) \cup V(P)$, which contradicts the maximality of $|V(C)|$ (see Figure 8). If l is even, then $D := u_l v^- \overleftarrow{C} v_{l-2}^+ u_{l-1} u_{l-2}\ v_{l-2} \overleftarrow{C} v_{l-4}^+ u_{l-3} u_{l-4} v_{l-4} \overleftarrow{C} \ldots v_2 \overleftarrow{C} u x \overrightarrow{P} x' v \overrightarrow{C} u_l$ is a cycle in G such that $V(D) = (V(C) \setminus \{u_1\}) \cup V(P)$, which contradicts the maximality of $|V(C)|$ again (see Figure 9). $\square$

Claim 7 *Let k be an integer with $2 \leq k \leq s-1$. If $u_1 \overleftarrow{C} u_{k-1}$ be an insertible path of C, then $u_{k+1} u_{k-2} \notin E(G)$, where $u_0 = u$.*

Proof. Let $v_1, \ldots, v_{k-1}$ be bridging vertices of $u_1 \overleftarrow{C} u_{k-1}$. Suppose that $u_{k+1} u_{k-2} \in E(G)$, and let $\overrightarrow{D} = u_{k+1} u_{k-2} \overrightarrow{C} v_{k-1} u_{k-1} u_k v_{k-1}^+ \overrightarrow{C} u_{k+1}$. Then D is a cycle in G such that $V(D) = V(C)$, and hence $\mu(D) = \mu(C)$ and $\omega(D) = \omega(C)$, in particular, H is also a component of $G - V(D)$. Since $N_G(H; w^+ \overrightarrow{C} u^- \cup u^+ \overrightarrow{C} v^-) = \emptyset$, it follows from the definition of D that u and v are distinct two vertices in D such that $N_G(u;H) \neq \emptyset$, $N_G(v;H) \neq \emptyset$, $|N_G(u;H) \cup N_G(v;H)| \geq 2$ and $N_G(H; u^+ \overrightarrow{D} v^-) = \emptyset$. Since $|V(u \overrightarrow{D} v)| > |V(u \overrightarrow{C} v)|$, this contradicts the choice (C3). $\square$

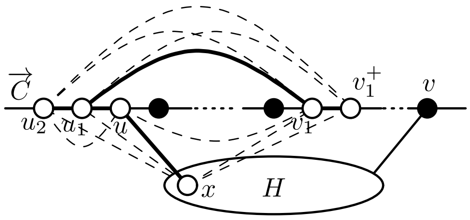

Figure 10: $G[\{x, u, u_1, u_2, v_1, v_1^+\}]$

Claim 8 *For each k with $1 \le k \le s-1$, $u_1 \overleftarrow{C} u_k$ is an insertible path of C.*

Proof. We first show that $u_1 \overleftarrow{C} u_1$ ($= u_1$) is an insertible path of C. Since $(u^- u^{+2} =)\ u_1 u^{+2} \in E(G)$ and $|V(u\overrightarrow{C}v)| \ge 6$, there exists a vertex v_1 in $N_G(u_1; u^{+2}\overrightarrow{C}v^{-2})$. We choose v_1 so that $|V(v_1\overrightarrow{C}v^{-2})|$ is as small as possible. By Lemma 3 (ii), (iii) and the choice of v_1, we have $v_1 \in V(u^{+2}\overrightarrow{C}v^{-3})$ and $N_G(u_1; v_1^+\overrightarrow{C}v^-) = \emptyset$. Suppose that $u_2 v_1^+ \notin E(G)$. Let $x \in N_G(u; H)$. Since $N_G(H; w^+\overrightarrow{C}u^- \cup u^+\overrightarrow{C}v^-) = \emptyset$ and G is K_3-free, we have $E(G) \cap \{xu_2, xu_1, xv_1, xv_1^+, u_2u, u_2v_1, uv_1\} = \emptyset$. Hence $G[\{x, u, u_1, u_2, v_1, v_1^+\}] \not\cong K_{1,3}^{**}$ yields that $uv_1^+ \in E(G)$ (see Figure 10).

Let $x' \in N_H(x)$. Recall that $v_1 \in V(u^{+2}\overrightarrow{C}v^{-3})$. By again the fact that $N_G(H; w^+\overrightarrow{C}u^- \cup u^+\overrightarrow{C}v^-) = \emptyset$ and G is K_3-free, we have that $E(G) \cap \{xu_1, xv_1^+, xv_1^{+2}, x'u_1, x'u, x'v_1^+, x'v_1^{+2}, uv_1^{+2}\} = \emptyset$. This together with the fact that $N_G(u_1) \cap V(v_1^+\overrightarrow{C}v^-) = \emptyset$ implies that $G[\{x, x', u, u_1, v_1^+, v_1^{+2}\}] \cong K_{1,3}^{**}$, a contradiction. Thus $u_2 v_1^+ \in E(G)$, and hence $u_1 \overleftarrow{C} u_1$ is an insatiable path of C.

We next show that for k with $2 \le k \le s-1$, $u_1 \overleftarrow{C} u_k$ is an insertible path of C. Suppose that there exists an integer k with $2 \le k \le s-1$ such that $u_1 \overleftarrow{C} u_k$ is not an insertible path of C. We choose k so that k is as small as possible. Then $u_1 \overleftarrow{C} u_{k-1}$ is an insertible path of C. Since $u_1 \overleftarrow{C} u_{k-1}$ is an insertible path of C, there exist bridging vertices $v_1, \dots, v_{k-1}$ of $u_1 \overleftarrow{C} u_{k-1}$. Note that by Claim 6, $v_{k-1} \in V(u^{+2}\overrightarrow{C}v^{-3})$, and hence by the condition (I3) and again Claim 6, we have

$$N_G(u_i; v_i^+\overrightarrow{C}v^-) = \emptyset \text{ for } 1 \le i \le k-1. \tag{4.1}$$

Since $v_{k-1} \in V(u^{+2}\overrightarrow{C}v^{-3})$, and since $u_k v_{k-1}^+ \in E(G)$ by the condition (I2), there exists a vertex v_k in $N_G(u_k; v_{k-1}^+\overrightarrow{C}v^{-2})$. We choose v_k so that $|V(v_k\overrightarrow{C}v^{-2})|$ is as small as possible. Then the choice of v_k implies that $N_G(u_k; v_k^+\overrightarrow{C}v^{-2}) = \emptyset$ if $v_k \ne v^{-2}$. Therefore, since $u_1 \overleftarrow{C} u_{k-1}$ is an insertible path of C and $u_1 \overleftarrow{C} u_k$ is not an insertible path of C, we have

$$u_{k+1} v_k^+ \notin E(G). \tag{4.2}$$

Since $u_1 \overleftarrow{C} u_{k-1}$ is an insertible path of C, it follows from Claim 7 that

$$u_{k+1} u_{k-2} \notin E(G), \tag{4.3}$$

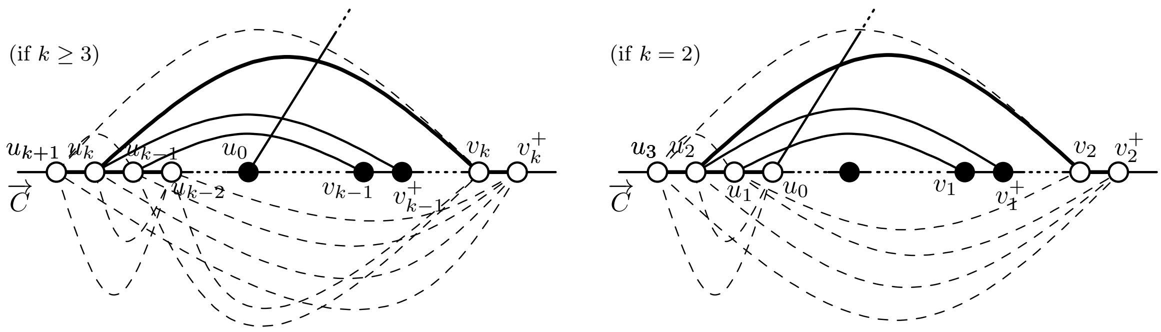

Figure 11: Claim 8

where $u_0 = u$. Since G is K_3-free, we also have that

$$E(G) \cap \{u_{k+1}u_{k-1}, u_{k+1}v_k, u_k u_{k-2}, u_k v_k^+\} = \emptyset. \tag{4.4}$$

If $k \geq 3$, then by combining (4.1)–(4.4), we have $G[\{u_{k+1}, u_k, u_{k-1}, u_{k-2}, v_k, v_k^+\}] \cong K_{1,3}^{**}$, a contradiction (see the left of Figure 11). Thus $k = 2$. Then by (4.1)–(4.4), and since $G[\{u_3, u_2, u_1, u, v_2, v_2^+\}] \not\cong K_{1,3}^{**}$, we have $E(G) \cap \{uv_2, uv_2^+\} \neq \emptyset$ (see the right of Figure 11).

Let $x \in N_G(u; H)$ and $x' \in N_H(x)$. If $uv_2^+ \in E(G)$, then since $N_G(H; w^+\overrightarrow{C}u^- \cup u^+\overrightarrow{C}v^-) = \emptyset$, it follows from (4.1) and (4.4) that $G[\{x, x', u, u_1, u_2, v_2^+\}] \cong K_{1,3}^{**}$, a contradiction. Thus $uv_2^+ \notin E(G)$, and hence $uv_2 \in E(G)$. Then since $N_G(H; w^+\overrightarrow{C}u^- \cup u^+\overrightarrow{C}v^-) = \emptyset$, it follows from (4.1) and (4.4) that $G[\{x, x', u, u_1, v_2, v_2^+\}] \cong K_{1,3}^{**}$, a contradiction. $\square$

By Claim 8, $u_1\overleftarrow{C}u_{s-1}$ is an insertible path of C. Let $v_1, \ldots, v_{s-1}$ be bridging vertices of $u_1\overleftarrow{C}u_{s-1}$. Note that $v_i \in V(u^{+3}\overrightarrow{C}v^{-2})$ for each i with $2 \leq i \leq s-1$. Let $x \in N_G(u; H)$ and $x' \in N_G(w; H)$ (if possible, choose $x' \neq x$), and let $\overrightarrow{P}$ be an (x', x)-path in H. Recall that $\{u_1u^{+2}, u_2u^+\} \subseteq E(G)$. If s is even, then $D := wx'\overrightarrow{P}xuu^+u_2u_1u^{+2}\overrightarrow{C}v_3u_3u_4\, v_3^+\overrightarrow{C}v_5u_5u_6v_5^+\overrightarrow{C}\ldots v_{s-1}u_{s-1}u_sv_{s-1}^+\overrightarrow{C}w$ is a cycle in G such that $V(D) = V(C) \cup V(P)$, a contradiction. Thus s is odd. Let $D = wx'\overrightarrow{P}xuu_1u^{+2}\overrightarrow{C}v_2u_2u_3v_2^+\overrightarrow{C}v_4u_4u_5v_4^+\overrightarrow{C}\ldots v_{s-1}\; u_{s-1}u_sv_{s-1}^+\overrightarrow{C}w$. Then D is a cycle in G such that $V(D) = (V(C) \setminus \{u^+\}) \cup V(P)$. Hence by the maximality of $|V(C)|$, $V(P) = \{x\}$, in particular, $w \neq v$. Moreover, if there exists a vertex y in $N_G(u^+; G - V(C))$, then by Claim 3, $yu^{+3} \in E(G)$, and hence $(D - \{u^{+2}u^{+3}\}) + \{u^+u^{+2}, yu^+, yu^{+3}\}$ is a longer cycle than C (note that by Lemma 3 (i), $y \notin V(H)$), a contradiction. Thus $N_G(u^+; G - V(C)) = \emptyset$. Therefore, u^+ is a component of $G - V(D)$, and $G - V(D)$ contains some components whose union is $H - \{x\}$. This implies that either $\mu(D) < \mu(C)$, or $\mu(D) = \mu(C)$ and $\omega(D) < \omega(C)$ holds, which contradicts the choice (C1) or (C2).

This completes the proof of Theorem 7. $\square$

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